

A Very Light Dilaton

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Abstract

We present a completely perturbative model that displays behavior similar to that of walking technicolor. In one phase of the model RG-trajectories run towards an IR-fixed point but approximate scale invariance is spontaneously broken before reaching the fixed point. The trajectories then run away from it and a light dilaton appears in the spectrum. The mass of the dilaton is controlled by the “distance” of the theory to the critical surface, and can be adjusted to be arbitrarily small without turning off the interactions. There is a second phase with no spontaneous symmetry breaking and hence no dilaton, and in which RG trajectories do terminate at the IR-fixed point.

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I. INTRODUCTION

The Nambu-Goldstone boson of spontaneously broken scale invariance is known as a dilaton. The name is also used to describe the pseudo Nambu-Goldstone boson, a massive state that appears when scale invariance is slightly broken. Classically this notion makes good sense. For example, take a scale invariant field theory, one with only dimensionless couplings,¹ with a flat direction for the minima of the potential for scalar fields. A dilaton follows from expanding about a non-zero field value. Adding arbitrarily small terms with dimensional couplings will generally give the dilaton a small mass. However, ordinarily the passage to the quantum case can destroy this picture. Quantum effects break scale invariance even in the absence of explicit mass terms. The state that before quantization would have been identified as a dilaton acquires a mass that is not small. In fact, it is not clear one can uniquely identify a state with what would have been the dilaton. What is meant by a “small” mass is that it can be made arbitrarily small while keeping all the remaining spectrum roughly constant and interacting. However it is not easy to construct models displaying this behavior, that is, models of a very light dilaton.

In their celebrated analysis of the massless abelian $U(1)$ model Coleman and Weinberg find a scalar of mass m and a vector of mass M in the spectrum, with $m^2/M^2 = 3e^2/8\pi^2$ [1]. Since the model is classically scale invariant one is tempted to identify the only scalar with the pseudo Nambu-Goldstone boson of broken scale invariance. It is not clear that this identification makes sense. But even if we insist on it we see that the dilaton can only be made light by turning off the interactions, $e^2 \rightarrow 0$. Moreover, if we insist in keeping the scale of symmetry breaking fixed then in this limit the vector meson mass also approaches zero, albeit at a slower rate.

One may guess that a good search strategy for a light dilaton model is to take

¹ In this work we consider only field theories in four space-time dimensions.

as a starting point an exactly conformal model. Then look to spontaneously break scale invariance and finally add small explicit scale symmetry breaking terms. But this strategy has proven ineffective. Consider, for example, $N = 4$ supersymmetric Yang-Mills theory, an exactly conformal interacting theory. The scalar potential has minimum energy flat directions and one can choose to expand about a non-trivial vacuum. Scale invariance is spontaneously broken and a massless dilaton must emerge. However, supersymmetry is not broken and a lot more massless stuff emerges too. As the vacuum breaks the Yang-Mills symmetry group G to one of its maximal subgroups H a full $N = 4$ H -gauge theory remains in the massless spectrum. The potential again has many zero energy flat directions and we are free to identify these with “dilaton.” Of course, we could just as well have identified with dilatons the flat directions of the original theory, based on G . Moreover, adding perturbations will render the dilaton very heavy, calling into question the identification of any one state with the dilaton. A perturbation, either relevant or marginal, vitiates the cancellations that give vanishing beta functions and the theory runs to strong coupling in the infrared.

In this work we construct a model of a light dilaton. The strategy, construction of the model and the results of our analysis are easily summarized. We look for a light dilaton in an interacting field theory that displays a perturbative attractive infrared fixed point and contains scalars. The idea is to look for spontaneous symmetry breaking along a renormalization group trajectory headed towards the fixed point. For a specific model we take that of Banks and Zaks [2] supplemented with scalars that are neutral under the gauge group. The scalars have quartic self-interactions and are Yukawa-coupled to the Banks-Zaks spinors. As the Yang-Mills gauge coupling runs toward the Banks-Zaks IR-fixed point, it drives the scalar and Yukawa couplings towards the non-trivial fixed point values too. Depending on the relative values of the coupling constants the Coleman-Weinberg effective potential for the scalar

fields may develop a non-trivial minimum [1]. The parameter space of the theory is split according to whether scaling symmetry is spontaneously broken or not, and for couplings near the boundary between these regions the dilaton is very light in units of its decay constant. Yet the theory is fully interacting and the spectrum is non-trivial (and insensitive to the parameter adjustment required to make the dilaton arbitrarily light).

Our search for a model of a very light dilaton was partially motivated by recent work of Appelquist and Bai [3] (henceforth ‘AB’) and by Hashimoto and Yamawaki [4] rekindling and old debate on whether walking technicolor (WTC) may have a light dilaton in its spectrum [5–7]. The idea of “walking” promises to solve many difficulties of technicolor (TC) theories. The conjectural behavior of the theory requires that (1) the TC coupling constant g evolves very slowly, (2) this occurs while at large value of the TC coupling constant, so that anomalous dimensions are large, and (3) the slowly running coupling eventually crosses a threshold, exceeding a critical value g_c for chiral symmetry breaking. The picture is that once the coupling crosses this threshold, techniquarks become massive, decouple and leave the technigluons to drive alone the running of the coupling constant (which from that point on grows quickly, much like in QCD). The condensate that results breaks electroweak symmetry giving masses to W and Z gauge bosons. The large anomalous dimension of the techniquark bi-linear insures that four-fermion operators induced by extended-TC interactions (ETC) give acceptable masses to all but the top quarks (and leptons) while effectively suppressing ETC mediated FCNCs. Moreover, the large anomalous dimensions of 4-techniquark operators also induced by the ETC tend to increase the masses of troublesome pseudo-Goldstone to acceptable levels. In this picture the slow evolution of the coupling constant can be viewed as an approach towards a would-be conformal fixed point, g_* . It is a “would-be” fixed point only because $g_c < g_*$, which triggers the fast QCD-like evolution of g once it exceeds the critical value g_c . AB ar-

gue, while Hashimoto and Yamawaki rebut, that a dilaton does appear and estimate that its mass is roughly determined by the value of the beta function at its closest approach to the would-be fixed point, $\beta(g_c)$.

The existence of a light dilaton in WTC is by no means obvious. The dilaton is in some respects similar to the η' in QCD. Were we to ignore the $U(1)_A$ anomaly it would be a pseudo Nambu-Goldstone boson, on par with the (π, K, η) octet. But the anomaly breaks the symmetry explicitly and because it involves the strong interactions this breaking is not a small perturbation. Beyond deciding whether the light dilaton appears in the spectrum of WTC, there are many other questions that arise. For example, what precisely is the meaning of the critical coupling g_c , what is the dilaton decay constant, etc.

Unfortunately, as of this writing there is no explicit realization of the WTC idea as a specific model. Numerous numerical studies are ongoing to determine whether QCD-like theories at the edge of the conformal window display the phenomenon of walking [8–19]. While a positive result from these studies may confirm the existence of models exhibiting the WTC idea, a negative result would not rule out the possibility that some non-QCD like theory behaves this way. In the mean time it would be useful to construct a toy model displaying some of the WTC behavior. One would like the toy model to be fully perturbative so that one may readily compute and resolve questions. In some ways our model fits the bill. It does have coupling constants that grow as they approach a fixed point, then walk for quite a long RG-time and finally swerve away. This change of behavior is triggered, much like in the WTC idea, by the analogue of chiral symmetry breaking, that is, the scalar fields acquiring a non-trivial expectation value, giving masses to the spinors through their Yukawa couplings. To be sure, the model fails to mimic WTC in important ways. By design it remains perturbative, and therefore anomalous dimensions remain small. And, as opposed to a would be WTC theory, our model is not asymptotically free; while the

Banks-Zaks sector is, RG-running in the scalar sector encounters Landau poles. We do not see the latter of these difficulties as central. One can view this as a theory with a cut-off at a scale that is exponentially large compared to where the physics of the symmetry breaking takes place, or imagine that it is the low energy effective theory of a more complete model.

But the usefulness of an explicit model of a very light dilaton goes beyond that of being a toy WTC. Sundrum has remarked that the dilaton can serve as a scalar analog of the graviton. By studying the properties of the dilaton one can hope to gain insights into the theory of gravity and perhaps find the answer to the cosmological constant puzzle [20]. A dilaton is also likely to appear in the AdS/CFT dual of the Randall-Sundrum model [21] with the Goldberger-Wise mechanism stabilizing the extra-dimension [22]. In the 4-dimensional language, the theory is described not by a CFT but by a flow to a CFT fixed point which is however interrupted close to the fixed point by the expectation value of a field that measures the distance from the origin in moduli space [23, 24]. This is described effectively by a theory at the fixed point, a CFT Lagrangian, supplemented by small perturbations. The latter are made scale invariant by including couplings to the dilaton in the spirit of phenomenological Lagrangians [25]. If the SM is embedded in such a scheme the dilaton may behave much like, but not exactly the same as, the higgs boson of the minimal standard model [26–28]. An amusing question that one can now ponder is the inverse AdS/CFT problem: given our perturbative model, what is the AdS dual (presumably a strongly interacting non-factorizable gravity model in 5 dimensions)?

Another area where the dilaton may play a role is in astrophysics and cosmology. By noting that the dilaton couples to the trace of the stress energy tensor, the authors in Ref. [29] propose to use a light dilaton as a force mediator between the SM particles and dark matter particles. Some authors also propose a light dilaton as a new dark matter candidate [30]. In all these cases an explicit computable model may be put

to use in understanding issues currently clouded by our inability to compute at or near strongly interacting fixed points.

The paper is organized as follows. In section 2 we introduce our model and show the existence of both the IR-fixed point and the non-trivial vacuum. In section 3 we identify the state corresponding to the dilaton and we compute its mass. In section 4 we discuss a phase structure of our model accessible in perturbation theory. We discuss our results briefly in Sec. 5.

II. THE MODEL

We study a class of $SU(N)$ gauge theories with $n_f = n_\chi + n_\psi = 2n_\chi$ flavors of spinors, ψ_i and χ_k , and two real scalars. The spinors are taken to be vector-like in the fundamental representation of the gauge group while the scalars are singlets. The most general Lagrangian that is classically scale invariant and also invariant under the discrete symmetry $\phi_1 \rightarrow \phi_1$, $\phi_2 \rightarrow -\phi_2$, $\psi \rightarrow \psi$, $\chi \rightarrow -\chi$, and the global simultaneous $SU(n_\chi)$ transformations $\psi \rightarrow U\psi$, $\chi \rightarrow U\chi$ is

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}\text{Tr } F^{\mu\nu}F_{\mu\nu} + \sum_{j=1}^{n_\chi} \bar{\psi}^j i \not{D} \psi_j + \sum_{k=1}^{n_\chi} \bar{\chi}^k i \not{D} \chi_k + \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 \\ & - y_1 (\bar{\psi}\psi + \bar{\chi}\chi) \phi_1 - y_2 (\bar{\psi}\chi + \text{h.c.}) \phi_2 - \frac{1}{24}\lambda_1 \phi_1^4 - \frac{1}{24}\lambda_2 \phi_2^4 - \frac{1}{4}\lambda_3 \phi_1^2 \phi_2^2. \end{aligned} \quad (1)$$

Quantum effects will induce scalar masses of the order of the cut-off. In the spirit of Coleman and Weinberg we happily subtract these masses away [1]; after all, we are not interested in solving the hierarchy problem. Alternatively one can study this theory perturbatively in the continuum, using dimensional regularization.

For small number of families this model is very similar to QCD. The gauge sector will run to strong coupling in the infrared, the remaining parameters will only act as small perturbations. The chiral symmetry $SU(n_f) \times SU(n_f)$ is spontaneously broken to its diagonal subgroup with associated Nambu-Goldstone bosons in the spectrum.

We are interested in larger values of n_f for which the gauge coupling is still asymptotically free but behaves very differently in the infrared, as we now discuss.

A. Fixed Point Structure

We arrange the values of N and n_f so that the coefficient in the one-loop term of the gauge beta function is small, much as Banks and Zaks do for QCD [2]. The perturbative fixed point value in the gauge coupling appears from balancing the one and two loop terms against each other. To arrange for an arbitrarily small fixed point value we consider only large values of N and n_f . The coefficients of the one-loop terms of the beta functions for the remaining couplings are not small. Hence it suffices to retain only up to one loop order in the beta functions of Yukawa and scalar couplings, while, of course, retaining up to two loop order for that of the gauge coupling. The mass independent (*e.g.*, minimal subtraction) β -functions at large N and n_f are given by [31]

$$\begin{aligned}
(16\pi^2)\frac{\partial g}{\partial t} &= -\frac{\delta N}{3}g^3 + \frac{25N^2}{2}\frac{g^5}{16\pi^2} \\
(16\pi^2)\frac{\partial y_1}{\partial t} &= 4y_1y_2^2 + 11N^2y_1^3 - 3Ng^2y_1, \\
(16\pi^2)\frac{\partial y_2}{\partial t} &= 3y_1^2y_2 + 11N^2y_2^3 - 3Ng^2y_2 \\
(16\pi^2)\frac{\partial \lambda_1}{\partial t} &= 3\lambda_1^2 + 3\lambda_3^2 + 44N^2\lambda_1y_1^2 - 264N^2y_1^4, \\
(16\pi^2)\frac{\partial \lambda_2}{\partial t} &= 3\lambda_2^2 + 3\lambda_3^2 + 44N^2\lambda_2y_2^2 - 264N^2y_2^4, \\
(16\pi^2)\frac{\partial \lambda_3}{\partial t} &= \lambda_1\lambda_3 + \lambda_2\lambda_3 + 4\lambda_3^2 + 22N^2\lambda_3y_1^2 + 22N^2\lambda_3y_2^2 - 264N^2y_1^2y_2^2.
\end{aligned} \tag{2}$$

The number of families is taken to be fixed at $n_f = 11N/2(1 - \delta/11)$ and we drop the $\mathcal{O}(\delta)$ terms except in β_g . Even though N and n_f are integers, one can make δ arbitrarily small by taking N and n_f arbitrarily large.

These equations will play an important role in our discussion. The first step is to determine whether any non-trivial fixed points exist. To see that one does indeed run into the fix point we can argue as follows. First, there is no question that the gauge coupling flows in the IR towards its fixed point. All that is required is that it starts its flow from the UV at a value smaller than the fixed point. Then the Yukawa couplings' beta functions are dominated by the last term, which is negative and only linear in the y_i 's. Hence they grow until the positive non-linear terms compensate against the last negative, linear term. And the story is then repeated for the scalars, but now having the Yukawa couplings drive the beta functions (the last terms in each of the three scalar coupling beta functions are negative and λ_i independent).

The mechanism that is driving the couplings towards the IR-fixed point values is mimicked by the process of determining their location. The gauge coupling has the same fixed point as in the Banks-Zaks model. This is used in the equations for the Yukawa couplings $y_{1,2}$ which are then trivially solved to leading order in $1/N$ accuracy. In turn these solutions are used in the equations for the scalar self-couplings. To leading order in $1/N$ accuracy, the fixed point is at the following zeroes of the beta functions:

$$g_*^2 = 16\pi^2 \frac{2}{75} \frac{\delta}{N}, \quad y_{1*}^2 = y_{2*}^2 = \frac{3}{11} \frac{g_*^2}{N}, \quad \lambda_{1*} = \lambda_{2*} = \lambda_{3*} = 6y_{1*}^2 = \frac{18}{11} \frac{g_*^2}{N}. \quad (3)$$

Since δ is arbitrarily small while N is arbitrarily large the fixed point values of the couplings are all perturbative. It is easy to check that the terms omitted in the loop expansion of the beta functions are parametrically smaller.

This result may be surprising. Common lore, which of course cannot be documented, is that theories with scalars and fermions do not exhibit nontrivial IR-fixed points in 4 dimensions. While this is obviously false in $4 - \epsilon$ dimensions, we see that it is also false in exactly four dimensions. The lore's intuition is vitiated here because it is the gauge coupling which is driving the remaining couplings toward the fixed

point.

B. Vacuum Structure

We turn now to the physical content of our model. The first order of business is to understand its vacuum structure and determine the fate of the symmetries of the Lagrangian. At the classical level, the potential is trivially minimized, $\langle\phi_1\rangle = \langle\phi_2\rangle = 0$ and all symmetries are explicitly realized. However, this may change once quantum effects are included. The one-loop effective potential in the $\overline{\text{MS}}$ scheme is [1]

$$\begin{aligned}
V_{\text{eff}} = & -\frac{1}{24}\lambda_1\phi_1^4 - \frac{1}{24}\lambda_2\phi_2^4 - \frac{1}{4}\lambda_3\phi_1^2\phi_2^2 \\
& - \frac{11N^2M_{f+}^4}{(64\pi^2)}\left(\ln\frac{M_{f+}^2}{2\mu^2} - \frac{3}{2}\right) - \frac{11N^2M_{f-}^4}{(64\pi^2)}\left(\ln\frac{M_{f-}^2}{2\mu^2} - \frac{3}{2}\right) \\
& + \frac{M_{s+}^4}{(64\pi^2)}\left(\ln\frac{M_{s+}^2}{\mu^2} - \frac{3}{2}\right) + \frac{M_{s-}^4}{(64\pi^2)}\left(\ln\frac{M_{s-}^2}{2\mu^2} - \frac{3}{2}\right),
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
M_{f\pm} &= y_1\phi_1 \pm y_2\phi_2, \\
M_{s\pm}^2 &= \frac{(\lambda_1 + \lambda_3)\phi_1^2 + (\lambda_2 + \lambda_3)\phi_2^2}{4} \\
&\quad \pm \frac{\sqrt{(\lambda_1 - \lambda_3)^2\phi_1^4 + (\lambda_2 - \lambda_3)^2\phi_2^4 - 2(\lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 - 7\lambda_3^2)\phi_1^2\phi_2^2}}{4}.
\end{aligned} \tag{5}$$

No mass terms have appeared because we have used dimensional regularization (in the $\overline{\text{MS}}$ scheme). As explained earlier, this is in keeping with Coleman and Weinberg who completely subtract the mass terms. We will return to this point in the discussion where we will argue that including small masses for the scalars and spinors of the model does not modify the main conclusions (but we have to wait until then to explain the meaning of “small.”)

It is fairly difficult to search for the minimum of this function. We can however find some local minima easily, by searching only for a vacuum that preserves the discrete symmetry $\phi_1 \rightarrow \phi_1$, $\phi_2 \rightarrow -\phi_2$, $\psi \rightarrow \psi$ and $\chi \rightarrow -\chi$. The effective potential along the $\phi_2 = 0$ axis is much simplified:

$$V_{\text{eff}} = \frac{\lambda_1}{24}\phi_1^4 + \frac{(\lambda_1\phi_1^2)^2}{256\pi^2} \left(\ln \frac{\lambda_1\phi_1^2}{2\mu^2} - \frac{3}{2} \right) + \frac{(\lambda_3\phi_1^2)^2}{256\pi^2} \left(\ln \frac{\lambda_3\phi_1^2}{2\mu^2} - \frac{3}{2} \right) - \frac{22N^2y_1^4\phi_1^4}{64\pi^2} \left(\ln \frac{y_1^2\phi_1^2}{\mu^2} - \frac{3}{2} \right). \quad (6)$$

It is straightforward to find an extremum of this function,

$$\begin{aligned} \frac{\partial}{\partial \phi_1} V_{\text{eff}}(\langle \phi_1 \rangle) &= 0 \\ \Rightarrow -\frac{\lambda_1}{6} &= \frac{\lambda_1^2}{64\pi^2} \left(\ln \frac{\lambda_1 \langle \phi_1 \rangle^2}{2\mu^2} - 1 \right) + \frac{\lambda_3^2}{64\pi^2} \left(\ln \frac{\lambda_3 \langle \phi_1 \rangle^2}{2\mu^2} - 1 \right) \\ &\quad - \frac{88N^2y_1^4}{64\pi^2} \left(\ln \frac{y_1^2 \langle \phi_1 \rangle^2}{\mu^2} - 1 \right). \end{aligned} \quad (7)$$

If the extremum is a minimum this equation determines the vacuum expectation $\langle \phi_1 \rangle$ in terms of the coupling constants of the model. Alternatively one may eliminate one of the dimensionless parameters of the model in favor of the dimensional vacuum expectation value. This is the well known dimensional transmutation procedure. Since the expectation value sets the physical scale for the theory we adopt this approach here so in what follows the dimensionless parameter λ_1 is understood as a function of the couplings and the expectation value, given in (7). In order that the perturbative expansion of V_{eff} not be invalidated by large logs in higher orders we insist that $\lambda_1/16\pi^2 \ln(\langle \phi_1 \rangle^2/\mu^2) \ll 1$. Then λ_1 is given by the last two terms in (7), and this condition becomes

$$\frac{\lambda_3^2 - 88N^2y_1^4}{(16\pi^2)^2} \ln^2 \frac{\langle \phi_1 \rangle^2}{\mu^2} \ll 1. \quad (8)$$

Since λ_1 has been eliminated in favor of $\langle \phi_1 \rangle$, the conditions that the perturbative analysis is valid are that $\langle \phi_1 \rangle$ satisfies (8) and that dimensionless couplings remain

small. Next, we must check that the extremum is a local minimum and that it is of lower energy than that of the origin of field space.

We first verify that the extremum is a local minimum. To this end we need to check that the eigenvalues of the mass matrix are both positive. Owing to the discrete symmetry and the fact that we are on the $\phi_2 = 0$ axis, the mixed derivatives terms vanish at $\langle\phi_1\rangle$, $\frac{\partial^2}{\partial\phi_1\partial\phi_2}V_{\text{eff}}(\langle\phi_1\rangle, 0) = 0$. Hence the two eigenvalues are given by

$$\frac{\partial^2}{\partial\phi_1^2}V_{\text{eff}}(\langle\phi_1\rangle, 0) = \frac{\lambda_3^2 - 88N^2y_1^4}{32\pi^2}\langle\phi_1\rangle^2, \quad (9)$$

$$\begin{aligned} \frac{\partial^2}{\partial\phi_2^2}V_{\text{eff}}(\langle\phi_1\rangle, 0) &= \frac{\lambda_3}{2}\langle\phi_1\rangle^2 - \frac{\lambda_3(\lambda_2 + 4\lambda_3)\langle\phi_1\rangle^2}{64\pi^2} \left(\ln \frac{\lambda_3\langle\phi_1\rangle^2}{2\mu^2} + 1 \right) \\ &\quad - \frac{264N^2y_1^2y_2^2\langle\phi_1\rangle^2}{64\pi^2} \left(\ln \frac{y_1^2\langle\phi_1\rangle^2}{\mu^2} - \frac{1}{3} \right). \end{aligned} \quad (10)$$

The first eigenvalue is positive provided

$$\varepsilon \equiv \lambda_3^2 - 88N^2y_1^4 > 0. \quad (11)$$

The second eigenvalue is generally positive provided we are in the regime where the one loop terms are small compared to the tree level term. This is generally the case in perturbation theory, although one could have one coupling, in this case λ_3 be small compared to the remaining couplings (and indeed this is the situation for λ_1 in the region of parameter space of interest).

We can now check that the effective potential at $\langle\phi_1\rangle$ is negative:

$$V_{\text{eff}}(\langle\phi_1\rangle) = -\frac{\lambda_3^2 - 88N^2y_1^4}{512\pi^2}\langle\phi_1\rangle^4 = -\frac{\varepsilon}{512\pi^2}\langle\phi_1\rangle^4. \quad (12)$$

Remarkably, the condition that this be negative is precisely the same as having the first eigenvalue of the mass matrix be positive, Eq. (11).

Here we take a small detour to discuss the role of ϕ_2 . The readers might notice that ϕ_2 plays virtually no role in the above analysis of the vacuum. Moreover, we

could have arranged for the non-trivial IR fixed-point with only one scalar. This can be seen by setting y_2 , λ_2 and λ_3 to zero in Eq (2) and repeating the fixed-point analysis given above.² This begs the question – what is the purpose of ϕ_2 ? With only one scalar, ϕ_1 , we can repeat the above analysis and reproduce Eqs. (8)–(12) with λ_3 set to 0. Clearly, the extremum becomes the maximum and the effective potential seems to be unbounded from below. The extra scalar field allow us to introduce more couplings and more importantly establish the non-trivial minimum via perturbative analysis.

Note that the conditions we have found for the non-trivial minimum of the effective potential are not satisfied at or in the vicinity of the IR-fixed point. But neither are the conditions for perturbative computability. In order to determine the vacuum structure near the IR fixed point we must re-sum the leading log expansion of the effective potential. Equivalently we can take any point in the vicinity of the fixed point and ask whether its RG-trajectory maps back at some large RG-time t to the region where the analysis above is valid. If that is the case we can further ask whether it gives a non-trivial minimum. This is the approach we adopt here. We will come back to this issue in Sec. IV where we will discuss the phase structure of the model and integrate the RGEs numerically to verify the vacuum structure near the IR-fixed point. But even without numerical studies we can argue physically that there are points arbitrarily close to the IR-fixed points for which the vacuum is non-trivial and scale invariance is spontaneously broken.

Choose the parameters to satisfy (11) and to be small at some fixed renormalization scale μ_0 . One can arrange for the allowed range of expectation values to be large, so that $\langle\phi_1\rangle \ll \mu_0$ is included, by choosing ε to be as small as necessary. The coupling constants will run as in the mass independent scheme until the scale

² Similar result regarding the fixed-point in Banks-Zaks type theory with an extra scalar singlet has been independently obtained in [32].

μ reaches values comparable to the mass of the heaviest particle in the model. At that point the running is modified. The trajectory that would end at the IR-fixed point is modified before the fixed point is reached. However this modification to the trajectory occurs only for $\mu \lesssim \langle \phi_1 \rangle$. That is, given a fixed starting point μ_0 we can choose to run as far as needed on the mass-independent trajectory, far enough that it gets arbitrarily close to the IR-fixed point; all that is required is that one starts with a small enough value of $\langle \phi_1 \rangle$.

We have not been able to explore fully the landscape of our effective potential. Other, lower minima may exist outside the $\phi_2 = 0$ axis. If that is the case the minimum we have found describes only a metastable vacuum. The analysis that follows is still largely correct. But more importantly, an analogous analysis could be applied to the global minimum and the qualitative results will not be different. What is important here is that the non-trivial minimum found at one-loop spontaneously breaks the scale invariance of the classical Lagrangian. The scale invariance is explicitly broken at one-loop too, by a quantum mechanical anomaly. If the former effect is dominant then we expect to see a pseudo Nambu-Goldstone boson of spontaneously broken approximate scale invariance, while if the latter effect is dominant no such state will be seen. So we turn in the next section to determining the spectrum of the model.

C. Particle Spectrum

If the theory is in the symmetric phase, $\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$, then all the particles are massless. Here, we compute the spectrum in the broken phase, $\langle \phi_1 \rangle = v$, $\langle \phi_2 \rangle = 0$. We retain up to one-loop order in the computation of the spectrum so that we may later address questions of invariance of physical quantities under RG-evolution. This is important because on the one hand we determine the vacuum structure far away

from the IR fixed point while on the other we are interested in the fate of scale invariance and hence want to study the RG flow towards, and eventually in the vicinity of, the IR-fixed point.

We first compute the fermion spectrum. For large N the leading contribution to the fermion self-energy is from the gauge interaction. We can parametrize the self-energy as

$$i\Sigma(\not{p}) = i(Am + B\not{p}). \quad (13)$$

We obtain, to one-loop order,

$$A = \frac{g^2}{16\pi^2} \frac{N}{2} \left(-3 \ln \frac{y_1^2 v^2}{\mu^2} + 4 \right), \quad B = 1, \quad (14)$$

in Landau gauge. Hence the masses of χ and ψ (poles in the respective propagators) are

$$M_\psi(\mu) = M_\chi(\mu) = y_1 v \left[1 - \frac{g^2}{16\pi^2} \frac{N}{2} \left(3 \ln \frac{y_1^2 v^2}{\mu^2} - 4 \right) \right]. \quad (15)$$

The pole masses of the scalar fields ϕ_1 and ϕ_2 can be computed in a similar manner. Schematically, to one-loop order, the mass is

$$M_\phi^2 = \frac{\lambda}{2} v^2 + \Pi(\lambda v^2/2). \quad (16)$$

Explicit computation yields

$$\begin{aligned}
M_{\phi_1}^2 &= \frac{\lambda_1 v^2}{2} + \frac{3\lambda_1^2 v^2}{64\pi^2} \left(\ln \frac{\lambda_1 v^2}{2\mu^2} - \frac{5}{3} + \frac{2\pi}{3\sqrt{3}} \right) + \frac{3\lambda_3^2 v^2}{64\pi^2} \left(\ln \frac{\lambda_3 v^2}{2\mu^2} - \frac{1}{3} - \frac{2\lambda_1}{3\lambda_3} \right) \\
&\quad + \frac{22N^2 y_1^2}{16\pi^2} \left[y_1^2 v^2 - \frac{\lambda_1 v^2}{12} - 3 \left(y_1^2 v^2 - \frac{\lambda_1 v^2}{12} \right) \left(\ln \frac{y_1^2 v^2}{\mu^2} \right) \right. \\
&\quad \left. - 3 \int_0^1 dx \left(y_1^2 v^2 - \frac{x(1-x)}{2} \lambda_1 v^2 \right) \ln \left(1 - x(1-x) \frac{\lambda_1}{2y_1^2} \right) \right], \quad (17) \\
&= \frac{3\lambda_1^2 v^2}{64\pi^2} \left(-\frac{2}{3} + \frac{2\pi}{3\sqrt{3}} \right) + \frac{3\lambda_3^2 v^2}{64\pi^2} \left(\frac{2}{3} - \frac{2\lambda_1}{3\lambda_3} \right) \\
&\quad + \frac{22N^2 y_1^2}{16\pi^2} \left[-2 \left(y_1^2 v^2 - \frac{\lambda_1 v^2}{12} \right) \right. \\
&\quad \left. - 3 \int_0^1 dx \left(y_1^2 v^2 - \frac{x(1-x)}{2} \lambda_1 v^2 \right) \ln \left(1 - x(1-x) \frac{\lambda_1}{2y_1^2} \right) \right],
\end{aligned}$$

$$\begin{aligned}
&\simeq \frac{\lambda_3^2 - 88N^2 y_1^4}{32\pi^2} v^2, \\
&= \frac{\varepsilon}{32\pi^2} v^2, \\
M_{\phi_2}^2 &= \frac{\lambda_3 v^2}{2} + \frac{\lambda_1 \lambda_3 v^2}{64\pi^2} \left(\ln \frac{\lambda_1 v^2}{2\mu^2} - 1 \right) + \frac{\lambda_2 \lambda_3 v^2}{64\pi^2} \left(\ln \frac{\lambda_3 v^2}{2\mu^2} - 1 \right) \\
&\quad + \frac{\lambda_3^2 v^2}{16\pi^2} \left(\ln \frac{\lambda_3 v^2}{2\mu^2} + \int_0^1 dx \ln \left(x^2 + (1-x) \frac{\lambda_1}{\lambda_3} \right) \right) \\
&\quad + \frac{22N^2 y_2^2}{16\pi^2} \left[y_1^2 v^2 - \frac{\lambda_3 v^2}{12} - 3 \left(y_1^2 v^2 - \frac{\lambda_3 v^2}{12} \right) \left(\ln \frac{y_1^2 v^2}{\mu^2} \right) \right. \\
&\quad \left. - 3 \int_0^1 dx \left(y_1^2 v^2 - \frac{x(1-x)}{2} \lambda_3 v^2 \right) \ln \left(1 - x(1-x) \frac{\lambda_3}{2y_1^2} \right) \right], \quad (18) \\
&\simeq \frac{\lambda_3 v^2}{2} + \frac{\lambda_2 \lambda_3 v^2}{64\pi^2} \left(\ln \frac{\lambda_3 v^2}{2\mu^2} - 1 \right) + \frac{\lambda_3^2 v^2}{16\pi^2} \left(\ln \frac{\lambda_3 v^2}{2\mu^2} - 2 \right) \\
&\quad + \frac{22N^2 y_2^2}{16\pi^2} \left[y_1^2 v^2 - \frac{\lambda_3 v^2}{12} - 3 \left(y_1^2 v^2 - \frac{\lambda_3 v^2}{12} \right) \left(\ln \frac{y_1^2 v^2}{\mu^2} \right) \right. \\
&\quad \left. - 3 \int_0^1 dx \left(y_1^2 v^2 - \frac{x(1-x)}{2} \lambda_3 v^2 \right) \ln \left(1 - x(1-x) \frac{\lambda_3}{2y_1^2} \right) \right].
\end{aligned}$$

The first lines of Eqs. (17) and (18) are the complete one-loop expressions for the pole masses, while the second line on Eq. (17) uses Eq. (7) and shows that the whole expression is of one-loop order and that it has no explicit μ dependence. The last line in both equations is further simplified using the approximation valid at μ_0 that λ_1 is small. Observe that these scalar masses differ from the curvature of the effective potential at the minimum. This is because the effective potential is computed at zero external momentum, while the pole mass is computed at a momentum equal to the pole mass itself.

It is instructive to check that these masses are RG-invariant. The important observation is that the vacuum expectation value, v , transforms under the RGE with the anomalous dimension of ϕ_1 :

$$\frac{\partial v}{\partial t} = \gamma_{\phi_1} v = -\frac{11N^2 y_1^2}{16\pi^2} v. \quad (19)$$

Using this, the above expressions for the pole masses and the beta functions in Eq. (2), one can verify that

$$\frac{\partial M_\psi}{\partial t} = \frac{\partial M_\chi}{\partial t} = \frac{\partial M_{\phi_1}}{\partial t} = \frac{\partial M_{\phi_2}}{\partial t} = 0, \quad (20)$$

up to terms of order of two loops. This is of course expected, but the explicit computation gives a check of the above expressions. For this check we have not used the approximation that λ_1 is small. This approximation is only valid for $\mu \sim \mu_0$, but we will be examining shortly RG-trajectories that extend to $\mu \ll \mu_0$ where the approximation breaks down.

III. DILATON

A. Dilatation Current

The dilatation current, \mathcal{D}^μ is related to the improved stress-energy tensor through $\mathcal{D}^\mu = x_\nu \Theta^{\mu\nu}$ [33]. There are two important properties of the improved energy momentum tensor. First, it is not renormalized, so it has no anomalous dimensions. And second, it is such that the divergence of the dilatation current is just the trace of the stress-energy tensor, $\partial_\mu \mathcal{D}^\mu = \Theta^\mu_\mu$. A simple way of computing this tensor is by re-writing the model in a general covariant fashion, with a background metric $g_{\mu\nu}$, taking $\Theta^{\mu\nu} = -2\frac{\delta}{\delta g_{\mu\nu}} S_m$ where S_m is the action integral (exclusive of the Hilbert-Einstein term) and then re-setting the metric to the trivial one $g_{\mu\nu} = \eta_{\mu\nu}$. From the Lagrangian in (1) we have

$$\begin{aligned} \Theta^{\mu\nu} = & -F^{a\mu\lambda} F_\lambda^{a\nu} + \frac{1}{2} \bar{\chi} i (\gamma^\mu D^\nu + \gamma^\nu D^\mu) \chi + \frac{1}{2} \bar{\psi} i (\gamma^\mu D^\nu + \gamma^\nu D^\mu) \psi \\ & + \partial^\mu \phi_i \partial^\nu \phi_i - \frac{1}{2} \kappa (\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi_i^2 - g^{\mu\nu} \mathcal{L}. \end{aligned} \quad (21)$$

The term proportional to κ is the improvement: it is automatically conserved and is itself a total derivative so its integral vanishes, leaving the generators of energy and momentum $\int d^3x \Theta^{0\mu}$ unmodified. The improved tensor corresponds to setting $\kappa = 1/3$.

Classically the trace of this tensor vanishes and therefore the divergence of the dilatation current vanishes too. The theory is classically scale invariant. As is famously known this is no longer the case once quantum effects are included. Instead one has a “trace anomaly:” [34, 35]

$$\Theta^\mu_\mu = \gamma_{\phi_1} \phi_1 \partial^2 \phi_1 + (4\gamma_{\phi_1} \lambda_1 - \beta_{\lambda_1}) \frac{\phi_1^4}{24} + \dots, \quad (22)$$

where we have kept only the terms involving ϕ_1 since these will play a role in our discussion below. The terms involving the field anomalous dimension γ_{ϕ_1} are often

overlooked. They can be ignored when application of the equations of motion is valid but may play a role in off-shell matrix elements or Green functions.³ There is a simple indirect indication that these additional terms must be included: since $\Theta^{\mu\nu}$ is not renormalized the trace anomaly must be an RG-invariant, and the γ_{ϕ_1} -terms are required for this purpose [37].

B. Dilaton

As a pseudo-Nambu-Goldstone boson the dilaton state $|\sigma\rangle$ should be created by acting on the vacuum with the spontaneously broken dilatation current. In analogy with PCAC we define a dilaton decay constant f_σ and a dilaton mass M_σ so that

$$\langle 0|\partial_\mu \mathcal{D}^\mu|\sigma\rangle = \langle 0|\Theta^\mu_\mu|\sigma\rangle_{x=0} = -f_\sigma M_\sigma^2. \quad (23)$$

This equation contains a particular combination of decay constant and mass and we would like to be able to distinguish between them. The matrix element of the current itself (which in PCAC gives the decay constant directly) is not very useful because of the explicit coordinate dependence. Instead consider the energy momentum tensor, before taking the trace:

$$\langle 0|\Theta^{\mu\nu}(x)|\sigma\rangle = \frac{f_\sigma}{3} (p^\mu p^\nu - g^{\mu\nu} p^2) e^{ip\cdot x} \quad (24)$$

The form of this equation is fixed by conservation of the stress-energy tensor and that its trace is given by Eq. (23). Note that in Eq. (24) the momentum is on-shell, $p^2 = M_\sigma^2$.

³ There is an interesting technical subtlety here. The equations of motion that can and should be used are those for the bare fields [36]. The use of the equation of motion in Eq. (22) gives that the terms proportional to γ_{ϕ_1} cancel. On the other hand, the insertion of the anomaly into a matrix element would have us replace $-M_{\phi_1}^2$ for ∂^2 but since this mass starts only at one-loop order its product with γ_{ϕ_1} would give a higher order effect and spoil the cancellation against the rest of the γ_{ϕ_1} terms. We have verified by explicit computation that in fact the cancellation is not spoiled. To this end one must use the relation in Eq. (7) that effectively trades λ_1 for one-loop terms.

In order to compute f_σ and M_σ we must first identify a state in the spectrum of our model as the dilaton. Were we in the exact symmetry limit there would be a unique one-particle state that couples to the stress energy tensor, making the identification of the dilaton straightforward. If the symmetry is not exact but approximate we expect the dilaton to be a spinless state that (1) couples most strongly to the stress energy tensor and (2) is the lightest state that does. It is easy to see that the state of mass M_{ϕ_1} fits the bill. First, it is the lightest of the two spinless one-particle states in the spectrum, which is clear since the perturbative expansion for its mass starts at one-loop order. To see that it couples more strongly, note that when expanding the fields about the vacuum $\langle\phi_1\rangle = v$ and $\langle\phi_2\rangle = 0$ in the stress energy tensor, the only field that appears linearly is ϕ_1 . Therefore the only one-particle state that has tree level overlap with the stress energy tensor is the state created by ϕ_1 .

With this identification we can now compute the decay constant to tree level. Shifting the fields in Eq. (21) and concentrating on terms that can give $p^\mu p^\nu$ in the matrix element, we have $\Theta^{\mu\nu} = -1/3v\partial^\mu\partial^\nu\phi_1 + \dots$. The ellipsis stand for terms that contribute only at higher order than tree level. Hence we read off $f_\sigma = v$. And, of course, $M_\sigma = M_{\phi_1}$.

The anomaly equation gives us a non-trivial check of this identification. Going to shifted fields in the anomaly Eq. (22), we have

$$\Theta_\mu^\mu = \gamma_{\phi_1}v\partial^2\phi_1 + (4\gamma_{\phi_1}\lambda_1 - \beta_{\lambda_1})\frac{v^3\phi_1}{6} + \dots \quad (25)$$

Taking the matrix element of this, working to lowest order (tree level in the graphs). we obtain

$$\langle 0|\Theta_\mu^\mu|\sigma\rangle_{x=0} = -\gamma_{\phi_1}vp^2 - \frac{\lambda_1^2 + \lambda_3^2 - 88N^2y_1^4}{32\pi^2}v^3 + \dots \quad (26)$$

This agrees with Eq. (23) if we use our identifications

$$f_\sigma = v \quad \text{and} \quad M_\sigma^2 = M_{\phi_1}^2 = \frac{\varepsilon}{32\pi^2}v^2. \quad (27)$$

We have dropped the $\gamma_{\phi_1} v M_\sigma^2$ and $\lambda_1^2 v^3$ terms for consistency.

Since the improved stress energy tensor is not renormalized the decay constant f_σ must be an RG-invariant quantity. M_σ is also RG-invariant as any physical mass must. The expressions we have found are not RG-invariant only because we have expressed them in lowest order of perturbation theory. The pole mass, which we have already discussed earlier, is explicitly seen to be RG-invariant to one-loop order for the trivial reason that it itself starts at one-loop order. On the other hand, the vacuum expectation value runs like the field, Eq. (19). If $Z(t)$ is the wave-function renormalization factor, $\partial Z/\partial t = 2\gamma_{\phi_1} Z$, $Z(0) = 1$, where $t = \ln(\mu/\mu_0)$, then $f_\sigma = v/Z^{1/2}$ is an RG-invariant, the RG-improved version of the previous result.

IV. PHASE STRUCTURE

We return here to the study of the phase structure of the model, posed earlier in Sec. IIB. Let us recapitulate from there. A perturbative study of the vacuum structure of the theory requires that we limit our attention to a region of parameter space where λ_1 is small. Then the model possesses a new, non-trivial minimum provided (11) is satisfied. Neither of these conditions are satisfied in the neighborhood of the IR-fixed point. However, we can take any point in the vicinity of the fixed point and ask whether its RG-trajectory maps back at some large RG-time t to the region where a perturbative analysis of the effective potential is valid and gives a non-trivial minimum. In fact, by reversing the process, that is, by starting with a well chosen point at large RG-time t and then running towards the IR, we argued that there always exist points arbitrarily near the IR-fixed point for which the symmetry is spontaneously broken. Choose coupling constants at some renormalization scale μ_0 that give a non-trivial minimum and so that the expectation value is small $\langle\phi_1\rangle \ll \mu_0$. The coupling constants will run as in the mass independent scheme to-

wards the IR-fixed point and will get closer the smaller the value of $\langle\phi_1\rangle$. At $\mu \sim \langle\phi_1\rangle$ the running will be modified and the trajectory will not hit the fixed point, but will have gotten very close.

Now let's complete the picture. When μ becomes of the order of the physical mass of the heaviest particles in the spectrum the running of the couplings is modified. For μ below the scale of that mass the beta function becomes effectively the one for the model in the absence of those massive particles, that is, the heavy particles are “integrated out.” As μ is further decreased one sequentially integrates out all massive particles in the model. This all occurs near the fixed point so all couplings are still perturbative, but now all scalars and spinors are integrated out. The Yukawa and self-couplings stopped running and become uninteresting since the effective theory contains only massless Yang-Mills vectors. Now the beta function of this effective theory is very much like that of QCD: the coupling constant quickly runs to strong coupling,

$$g^2(\mu) \approx \frac{g_*^2}{1 + \frac{g_*^2}{16\pi^2} \frac{22N}{3} \ln \frac{\mu}{\langle\phi_1\rangle}} \quad (28)$$

The spectrum of the effective theory is that of a theory of pure glue, that is glueballs, of mass

$$M_g \sim \langle\phi_1\rangle e^{-\frac{3}{22N} \frac{16\pi^2}{g_*^2}} = \langle\phi_1\rangle e^{-225/44\delta} \quad (29)$$

So the spectrum of the model consists of two massive scalars and n_f massive fermions with masses given in Sec. II C plus glueballs with masses M_g . The lighter scalar can be identified with the dilaton and its mass is given by Eq. (27).

We can repeat the analysis, only now starting from a set of coupling constants that does not satisfy the condition (11) at μ_0 . The potential now remains positive up to large values of ϕ_1/μ_0 and one expects that by the time it starts decreasing perturbation theory ceases to be applicable. So we expect the true vacuum is at the origin of field space $\langle\phi_1\rangle = \langle\phi_2\rangle = 0$. There is no spontaneous scaling symmetry

breaking, all particles are massless. As $t \rightarrow -\infty$ the RG-trajectories run into the IR-fixed point.

The following picture emerges: the theory has two phases. The parameter space of the model, which we identify with the space of couplings at a fixed renormalization scale μ_0 , is split in two regions. In region I the spectrum is massless and all RG-trajectories run into the IR-fixed point. In region II there are no massless particles and RG-trajectories do not end at the IR-fixed point. There is a boundary between these phases, a hypersurface in the parameter space of the model. The fixed point lies on this surface.

The expectation value $\langle\phi_1\rangle$ vanishes in region I, but does not in region II. The transition is discontinuous: by dimensional transmutation, there is a non-trivial minimum of V_{eff} at an arbitrary⁴ value of $\langle\phi_1\rangle$ provided $\lambda_3^2 - 88N^2y_1^4$ is positive, no matter how small. Since the physical content is preserved by flows we see that the surface itself is RG-invariant.

But perhaps we have rushed into conclusions. Firstly, when (11) is not satisfied the effective potential is unbounded from below as one moves along the ϕ_1 axis towards large values of ϕ_1 . We stated without justification that at large ϕ_1 perturbation theory breaks down and one expects the potential stays bounded from below. But there is no guarantee of this, and even if the potential stays bounded it may develop a new global minimum at large ϕ_1 . Perhaps none of region I is physical? And secondly, in order to reach the vicinity of the IR point, which is AB's prescription for obtaining a light dilaton, we argued we can choose $\langle\phi_1\rangle$ small enough that our RG-trajectory will get there. But how do we know that this does not occur only for such small $\langle\phi_1\rangle$ that the logs in the effective potential become too large, again invalidating the analysis?

⁴ Arbitrary, but not extreme: the logs of $\langle\phi_1\rangle/\mu_0$ cannot be too large if the perturbative analysis is to remain valid.

Fortunately we can go a long way towards settling these issues by explicit computation. Inasmuch as the potential becomes one dimensional (the minimum or the unbounded direction both lie on the axis) we can use the RGE to re-sum the leading logs hence extending the region of validity of the computation to the whole space of perturbative parameters. For the effectively one dimensional case the effective potential is $V_{\text{eff}} = \frac{1}{24} \bar{\lambda}_1(t, \lambda_1) Z(t)^2 \phi_1^4$ [38]. Here $t = \ln(\phi_1/\mu_0)$, Z is a wave-function renormalization factor and $\bar{\lambda}_1(t, \lambda_1)$ is the running coupling constant, defined with boundary condition $\bar{\lambda}_1(0, \lambda_1) = \lambda_1$. The first objection above is settled as follows: for any RG-trajectory for which $\bar{\lambda}_1$ stays positive we can assert the minimum of V_{eff} is at the origin of field space and there is no symmetry breaking. The only caveat is that we cannot trust the calculation at very large t where the scalar couplings become non-perturbatively large. Recall the model has Landau poles so it either is considered as a cut-off model or as the low energy limit of a complete theory.

The second objection can also be settled by following the trajectory towards the IR. If at any point along the trajectory the running coupling turns negative then there will be a minimum away from the origin in field space, symmetry will be broken and a pseudo Nambu-Goldstone boson associated with the breaking of scale invariance will appear in the spectrum. One can then follow the trajectory and determine how close it gets to the IR-fixed point. This is somewhat unnecessary, since we already established in the previous two sections that for small ε we get a light dilaton.

Although the model is perturbative, we do not know how to analytically integrate the RG trajectories. But it is quite straightforward to investigate them numerically. It is beyond the scope of this work to conduct an exhaustive study of the phase diagram numerically. Instead we follow the trajectories from some initial points at μ_0 to gain confidence the picture we have painted is not obviously flawed. We use $N = 20$, $n_f = 11N/2$, $\delta = 0.2$. First we take $g(\mu_0) = \frac{4}{9}g_*$, $y_1(\mu_0) = 0.45y_{1*}$, $y_2(\mu_0) = \frac{1}{5}y_{2*}$, $\lambda_1(\mu_0) = \frac{1}{30}\lambda_{1*}$, $\lambda_2(\mu_0) = 3\lambda_{2*}$, $\lambda_3(\mu_0) = 5.2\lambda_{3*}$. This set of parameters does

not satisfy (11). The effective potential doesn't develop a non-trivial minimum, the running coupling $\bar{\lambda}_1$ remains positive. The theory flows to the IR fixed point. Next we analyze the case when $g(\mu_0) = \frac{4}{9}g_*$, $y_1(\mu_0) = 0.32y_{1*}$, $y_2(\mu_0) = \frac{1}{5}y_{2*}$, $\lambda_1(\mu_0) = \frac{1}{30}\lambda_{1*}$, $\lambda_2(\mu_0) = 3\lambda_{2*}$, $\lambda_3(\mu_0) = 5.2\lambda_{3*}$. Naively, this theory seems to flow to the IR-fixed point as well. But in this case, the effective potential does develop a minimum at $\ln(v^2/\mu_0^2) \simeq -58$. We estimate the fractional correction to the effective potential from higher orders in the loop expansion to be of order

$$\left| \frac{Ng^2}{16\pi^2} \ln \left(\frac{y_1^2 v^2}{\mu^2} \right) \right| \simeq 0.2$$

Thus we can trust the minimum we find using perturbation theory. With this vev, the spectrum is $M_{\psi, \chi}/v \simeq 8.5 \times 10^{-3}$, $M_{\phi_1}/v \simeq 7.9 \times 10^{-4}$, $M_{\phi_2}/v \simeq 9.5 \times 10^{-2}$. The scale μ_0 is some 13 orders of magnitude larger than the vacuum expectation value v , but it is unphysical.

We have studied numerically the transition between these two parameters sets by varying $y_1(\mu_0)$ or $\varepsilon(\mu_0)$. When $y_1(\mu_0)$ is sufficiently large, or when ε turns negative, we change from a broken phase to the symmetric phase as expected from Eq. (12). Note that with our particular value of parameters, the theory is close to the boundary of the broken/symmetric phases.

A. Relevant Perturbations

Suppose we consider a modification of the model, one in which scale invariance is explicitly broken. This is accomplished by adding relevant perturbations. If the symmetries of the model are to be preserved only mass terms can be added. This enlarges the parameter space of the model. The origin of all the relevant-perturbation axes corresponds to the parameter space described in the previous paragraphs, and it is on that hyperplane that the IR-fixed point lies together with the two phases and

the hypersurface separating them.

Far away from this hyperplane, a long ways along the relevant-perturbation axes, the physics is very simple: scalars and spinors have hard masses and below the scale of those masses they decouple so as to leave only light glueballs in the spectrum. A more interesting region of parameter space is the direction of large scalar masses and small spinor masses. Then the scalars decouple and one is left with a Banks-Zaks-like model. Only it does not run into an IR-fixed point because the spinors eventually decouple, the YM-coupling then runs strongly and glueballs form. Only at zero spinor mass do we see that our IR-fixed point is really part of an IR-fixed hyperline.

What is the fate of the two phases as one extends into the new axes? In the symmetric phase the addition of hard masses can only make the vacuum at the origin of field space more stable. The spectrum is modified, particles are massive now and there is no IR-fixed point (save for the zero spinor mass case).

Analysis of the broken symmetry phase is more subtle. Provided we stay very close to the origin of the new axes, so that the added mass terms are really small perturbations, much smaller than the masses obtained in the absence of the perturbations, then nothing changes qualitatively and the quantitative changes to the spectrum are small. As the strength of the relevant perturbations increase the model may remain in a broken phase, depending on the precise nature of the perturbations. But for large enough perturbations the dilaton will be unrecognizable as a pseudo Nambu-Goldstone boson.

Summarizing, the two phase diagram does extend into the larger parameter space. The fixed point becomes a (hyper) line of fixed points. For large perturbations the dilaton is gone.

V. DISCUSSION, CONCLUSION AND OPEN QUESTIONS

We have presented a model with an IR-fixed point, and demonstrated that the model has two phases. In phase I RG-trajectories run into the IR-fixed point (in infinite RG-time). The scale symmetry is approximate and explicitly realized and it becomes exact at the fixed point. In phase II scale symmetry is spontaneously broken. Of course, scale invariance is also explicitly broken by the trace anomaly. The trajectories don't reach the IR-fixed point but some get very close and for those the explicit, relative to spontaneous, breaking of scale invariance is small: A light dilaton appears in the spectrum.

Analytic evidence for this picture was presented at length but the numerical support was scant. This is clearly an interesting direction for future work. In particular, one could determine the actual location of the phase transition. Another direction for future work is to find generalizations of the model. We do not know how general this picture is or how difficult it may be to come about models that display arbitrarily light dilatons (we were not aware of any example prior to this work).

Among new models one may try to construct some with the Standard Model of electroweak interactions embedded in it. One could then test whether the setup in Ref. [26] works as advertised. The authors there considered the possibility that the standard model is embedded in an almost conformal, possibly strongly interacting field theory with spontaneously broken scale invariance. In the context of 4-dimensional strongly interacting near-CFTs obtained as AdS/CFT-like duals of 5-dimensional non-factorizable geometries (RS models) one encounters often the schematic Lagrangian describing the dynamics:

$$\mathcal{L} = \mathcal{L}_{\text{CFT}} + \sum_n \lambda_n \mathcal{O}_n . \quad (30)$$

The first term is a CFT while the sum that follows is an attempt to capture the

deviations (“deformations”) from the CFT by adding small perturbations [23, 24]. Obviously this basic setup applies to our model, and because it is fully perturbative model one should be able to verify the validity of some general assertions. The deviations from conformality can be small in one of two ways, either the anomalous dimensions γ_n or the coefficients λ_n of the operators \mathcal{O}_n are small. On general grounds one can show that for $|\gamma_n| \ll 1$ the effective potential for the field χ whose expectation value gives rise to the dilaton is [26]

$$V_{\text{eff}}(\chi) = \frac{M_\sigma^2}{4f_\sigma^2} \chi^4 \left[\ln \left(\frac{\chi}{f_\sigma} \right) - \frac{1}{4} \right] + \mathcal{O}(\gamma^2). \quad (31)$$

The case $|\lambda_n| \ll 1$ is more cumbersome. Only in the case that only one perturbation is added does one obtain a parameter-free effective potential

$$V_{\text{eff}}(\chi) = \frac{M_\sigma^2}{f_\sigma^2 \gamma} \chi^4 \left[\frac{1}{4 + \gamma} \left(\frac{\chi}{f_\sigma} \right)^\gamma - \frac{1}{4} \right] + \mathcal{O}(\lambda^2),$$

while for more than one perturbation occur one has the less restricted

$$V_{\text{eff}}(\chi) = \frac{M_\sigma^2}{f_\sigma^2} \chi^4 \sum_n \left\{ x_n \left[\frac{1}{4 + \gamma_n} \left(\frac{\chi}{f_\sigma} \right)^{\gamma_n} - \frac{1}{4} \right] \right\} + \mathcal{O}(\lambda^2),$$

where the coupling constants have been traded for constants x_n that are constrained by $\sum_n \gamma_n x_n = 1$.

Any model with a conformal fixed point g_* can be written in the fashion of Eq. (30)

$$\mathcal{L}(g) = \mathcal{L}(g_*) + (\mathcal{L}(g) - \mathcal{L}(g_*))$$

where g are coupling constants at arbitrary values. If g is sufficiently close to g_* one is in the case $|\lambda_n| \ll 1$ above, while if the region of couplings that includes g and g_* is perturbative one expects $|\gamma_n| \ll 1$. We need, in addition, that the model display spontaneous breaking of scale invariance in the vicinity of the fixed point. Our model furnished an explicit example. The analogue of χ is our field ϕ_1 . Because

it is perturbative one has $|\gamma_n| \ll 1$. Reassuringly, when the tree level term in the effective potential of Eq. (6) is eliminated by use of Eq. (7), and the expressions for dilaton mass and decay constant in Eq. (27) the resulting potential is *exactly* of the form of Eq. (31). To emphasize, the dependence on the many coupling constants of our model is completely contained now in only two parameters: M_σ and f_σ .

Finally, we address one of the central questions we set out to investigate: Is the AB estimate of the dilaton mass in walking technicolor scenarios correct? For AB, the dilaton mass is given by

$$M_\sigma^2 \simeq \frac{s(\alpha_* - \alpha_c)}{\alpha_c} \Lambda^2 \simeq \frac{N_f^c - N_f}{N_f^c} \Lambda^2, \quad (32)$$

where α_* is the coupling at the fixed point, N_f is the number of flavors and Λ is the scale of chiral symmetry breaking which occurs only if the critical coupling α_c is below the fixed point, $\alpha_c < \alpha_*$, which in turn corresponds to the number of flavors below a critical value, N_f^c . The middle expression in Eq. (32), relating the mass to the distance between the critical coupling and the fixed-point, does not carry over to our model. In our case, the role of the critical value of the coupling constant α_c is played by a critical surface, $\varepsilon = 0$, separating the symmetric and broken phases. But the mass of the dilaton is not proportional to the distance between this surface and the fixed point (however one defines distance): the fixed-point lies on the critical surface and the dilaton mass vanishes everywhere on the surface. The rightmost expression in Eq. (32), however, has a counterpart in our model. In that formula $(N_f^c - N_f)/N_f^c$ measures how far the theory is from the critical point. In our model ε plays the role of this quantity. It measures how far the theory is from the critical surface. Moreover, both $(N_f^c - N_f)/N_f^c$ and ε can be made arbitrarily small which in turn make the dilaton arbitrarily light compared to the scale of symmetry breaking. To the extent that one can arrange for arbitrarily small $(N_f^c - N_f)/N_f^c$, AB's estimate of a parametrically small dilaton mass is consistent with our analysis.

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